Section 15.7

Optimization

The definitions of extrema follow naturally those found in Calculus 1.

Definition: Local Extrema
Suppose (a, b) is a point in a region R on which f is defined. If $f(x, y) \leq f(a, b)$ for all (x, y) in the domain of f and in some open disc centered at (a, b), then $f(a, b)$ is a **local maximum** of f.

Suppose (a, b) is a point in a region R on which f is defined. If $f(x, y) \geq f(a, b)$ for all (x, y) in the domain of f and in some open disc centered at (a, b), then $f(a, b)$ is a **local minimum** of f.

We may also directly extend Fermat's Theorem for Local Extrema:

Theorem 15.14:
If f has a local maximum or minimum value at (a, b) and the partial derivatives f_x and f_y exist at (a, b), then $f_x(a, b) = f_y(a, b) = 0$.

The points identified in Fermat's Theorem are **critical points**.

Definition:
An interior point (a, b) in the domain of f is a **critical point** of f if

1. $f_x(a, b) = f_y(a, b) = 0$, or
2. at least one of f_x or f_y does not exist at (a, b).

If f is differentiable at a critical point (a, b) and in every open disc centered at (a, b), there are points (x, y) such that $f(x, y) > f(a, b)$ and points for which $f(x, y) < f(a, b)$, then f has a **saddle point** at (a, b).

Example 15.7.1. Find the critical points of $f(x, y) = xy(x - 2)(y + 3)$.

The Hessian and Second Derivative Test

More specialized than the comparable result in Calculus 1, the concavity of a multivariable function is measured by the **Hessian matrix**

$$H_f = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}.$$

The **discriminant** of a function f with continuous second partial derivatives is the determinant of the Hessian

$$\det H_f = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2.$$

Theorem 15.15: Second Derivative Test
Suppose the second partial derivatives of f are continuous throughout an open disc centered at the point (a, b) where $f_x(a, b) = f_y(a, b) = 0$.

1. If $\det H_f(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum value at (a, b).
2. If $\det H_f(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum value at (a, b).
3. If $\det H_f(a, b) < 0$, then f has a saddle point at (a, b).
4. If $\det H_f(a, b) = 0$, then the test is inconclusive.

Definition:
Let f be defined on a set R in $\mathbb{R}^2$ containing the point (a, b). If $f(a, b) \geq f(x, y)$ for every (x, y) in R, then $f(a, b)$ is an **absolute maximum value** of f on R. If $f(a, b) \leq f(x, y)$ for every (x, y) in R, then $f(a, b)$ is an **absolute minimum value** of f on R.

We note that the Extreme Value Theorem extends to functions of n independent variables. That is, functions that are continuous on a closed and bounded set K obtain an absolute maximum and absolute minimum on the set K.

Procedure to Find Absolute Extrema on Closed Bounded Sets
Let f be continuous on a closed and bounded set K in $\mathbb{R}^2$. To find the absolute maximum and minimum values of f on K:

1. Determine the values of f at all critical points in K.
2. Find the maximum and minimum values of f on the boundary of K.
3. The greatest function value found above is the absolute maximum value of f on K. The least function value found above is the absolute minimum value of f on K.

Example 15.7.2. Find the absolute extrema of $f(x, y) = \sqrt{x^2 + y^2 - 2x + 2}$ over the region $K = \{(x, y) : x^2 + y^2 \leq 4, y \geq 0\}$.

Example 15.7.3. Find the absolute maximum and minimum values of $f(x, y) = 4 - x^2 - y^2$ on the open disc $R = \{(x, y) : x^2 + y^2 < 1\}$ (if they exist).$\epsilon, \varepsilon, \in$